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THE SUBGRAPH HOMEOMORPHISM PROBLEM FOR SMALL WHEELS

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The subgraph homeomorphism problem, where the pattern graph is a wheel with four or five spokes, is studied. The most important result is that any 3-connected graph satisfying a simple edge connectivity condition has a W_5-homeomorph iff it has a vertex v of degree at least 5 and a circuit, disjoint from v, of length at least 5. Efficient algorithms are described for these cases of the subgraph homeomorphism problem.

1. Introduction

A graph G is said to be an *elementary subdivision* of a graph H if an isomorphic copy of G can be obtained from H by replacing some edge of H with a path of length 2 (where the extra vertex so introduced does not occur in H). G is a *subdivision* of H if G can be obtained from H by a sequence of elementary subdivisions. Here we often say simply that G is a *H-subdivision*. The equivalent expression "G is homeomorphic from H" is also found in the literature.

Many authors have considered the structure of graphs containing no subdivision of another fixed graph (often called the *pattern graph*). Elegant characterizations have been obtained when the fixed graph is K_3, K_4[1, 2], $K_{3,3}$[4], C_4[9], C_5[9], and the two vertex graph with k parallel edges (for any k) [7], among others. It is our purpose here to add the wheels with four and five spokes to this list. The result on W_4 is relatively easy, although I have not been able to find it in the literature; the result on W_5 is more difficult.

We are also interested in algorithms for detecting the presence of various subdivisions. Specifically, if H is a fixed graph then we would like to know the complexity of the following problem.

SUBGRAPH HOMEOMORPHISM (H) (abbreviated SHP(H)).
Input: Graph G.
Question: Does G contain a subgraph which is a subdivision of H?

* The work of this paper was done while the author was a graduate student at the Mathematical Institute, Oxford, U.K.

It clearly belongs to NP for all H, and it is well known that if H is allowed to vary, as part of the input to the problem, then the problem becomes NP-complete. (We refer the reader to [3] for complexity theoretic terminology.) One may ask: for which fixed graphs H is SHP(H) in P and for which graphs is it NP-complete? This question was first asked by LaPaugh and Rivest [6]. For the pattern graphs listed above, the known elegant characterizations yield efficient polynomial time algorithms for the corresponding subgraph homeomorphism problems. In general, however, the question remained open until recently, when Robertson and Seymour used very deep and powerful techniques to prove that SHP(H) belongs to P for all H (see [8]).

Thus, in the sense of polynomial time solvability versus NP-completeness, the subgraph homeomorphism problem is now completely solved. The polynomial time algorithms found by Robertson and Seymour are very general; however they are not practical and do not yield elegant characterizations of the sort, for example, of those described above. Thus it is still of interest to obtain exact characterizations of graphs with subdivisions of a specific pattern graph excluded, and efficient algorithms for SHP(H) for specific graphs H. It is with this motivation that we study SHP(W_4) and SHP(W_5). It should perhaps be added that such results for specific pattern graphs do not in any way supersede, or compare with, the truly marvellous general results of Robertson and Seymour. They merely refine our understanding of a few special cases.

2. Some definitions

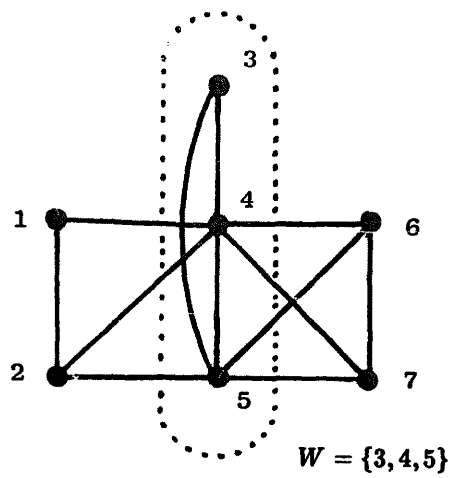
All graphs in the rest of this paper are undirected, and have no loops or multiple edges. Our notation and terminology follows standard usage; see e.g. [5] for undefined terms.

If G is a graph we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The *neighbourhood* $N_G(v)$ of a vertex v in G is the set of vertices which are adjacent to v in G. If $V' \subseteq V(G)$ then the subgraph of G *induced* by V' is (V', E') where $E' = \{vw \in E(G) \mid v, w \in V'\}$ and is denoted by $\langle V' \rangle$.

Paths are assumed to be self-avoiding. The *length* of a path or circuit is the number of edges it has. W_k denotes the *k-wheel* with k spokes (rather than $k-1$ spokes as in [5]). A vertex of a path P is *internal* if it is not an endpoint of P. If P

Table 1. Interval notation for subpaths.

Symbol	Symbol denotes the subpath of P between v and w...
$P(v, w]$	... excluding v, including w
$P[v, w)$	... including v, excluding w
$P[v, w]$	... including v and w



Bridges of $G|W$:
$\{1,2,4,5\}$;
$\{4,5,6\}$;
$\{4,5,7\}$;
$\{3,4,5\}$.

Fig. 1. A graph G, and the bridges of $G \mid W$.

is any path and v, $w \in V(P)$ then we use interval notation, as shown in Table 1, to describe various subpaths of P. The table gives full details. (The empty "path" with no vertices may result in some cases.)

If $W \subseteq V(G)$ then $G-W$ is the graph obtained from G by removing all vertices of W and all edges incident with vertices in W. If H is a subgraph of G then $G-H$ denotes the graph $G-V(H)$.

We say that $W \subseteq V(G)$ is a *separating set* of G, and that W *separates* G, if $G-W$ is disconnected. Thus, G has a separating set of size $\leq k$ if and only if G is not $(k+1)$-connected. If W separates G then we denote by $G \mid W$ the set of all subsets U of $V(G)$ satisfying

(i) Any two vertices of U are joined by a path in G with no internal vertex in W;

and

(ii) U is maximal with respect to (i).

We will refer to the elements of $G \mid W$ as *bridges*. Note that in this paper bridges are sets of vertices, not subgraphs. It is easily seen that every vertex of G lies in a bridge of $G \mid W$. Furthermore, the sets of the form $U \setminus W$, where U is a bridge of $G \mid W$, partition $V(G) \setminus W$.

Note that if G is disconnected and $W = \emptyset$ then the bridges of $G \mid W$ are precisely the vertex sets of the connected components of G.

An example illustrating the above definitions is given in Fig. 1.

3. Characterizations

We consider the structure of graphs with, in turn, W_4-subdivisions and W_5-subdivisions excluded. Most of this section is taken up with Theorem 4 on W_5.

The first result characterizes 3-connected members of SHP(W_4). It is reminiscent of Tutte's characterization of 3-connected graphs (see [5, p. 46]) but we cannot see a more direct connection.

Theorem 1. *Let G be a 3-connected graph. Then G contains a W_4-subdivision if and only if G has a vertex of degree greater than or equal to 4.*

We will prove a technical strengthening of this result which we will need for Theorem 4.

Definition. If $k \geqslant 4$ and H is a subgraph of G which is a W_k-subdivision, then we say H is *centred* on a vertex $v \in V(G)$ if v has maximum degree in H.

Lemma 2. *If G is a 3-connected graph and v is any vertex of degree $\geqslant 4$ in G, then G contains a W_4-subdivision centred on v.*

Proof. Suppose G is 3-connected and that $v_0 \in V(G)$ with $\deg v_0 \geqslant 4$. Let v_1, v_2, v_3, v_4 be four neighbours of v_0. By the 3-connectivity of G there exist paths P_1 and P_2 from $\{v_1, v_3\}$ to $\{v_2, v_4\}$ in G which are vertex-disjoint and avoid v_0. Now consider the vertex sets of these paths. Again by the 3-connectivity of G there exist two vertex-disjoint paths Q_1 and Q_2 from $V(P_1)$ to $V(P_2)$ each of which meets each of $V(P_1)$ and $V(P_2)$ only once and avoids v_0. These paths P_1, P_2, Q_1, Q_2 together with v_0 and the edges v_0v_i, $1 \leqslant i \leqslant 4$, constitute the required W_4-subdivision. □

We now attack the wheel W_5. A technical definition is necessary.

Definition. An *internal 3-edge-cutset* in a graph G is a set E' of at most 3 edges of G such that $G-E'$ is disconnected with each component having more than one vertex.

Theorem 3. *Let G be 3-connected, with no internal 3-edge-cutset. Then G has a W_5-subdivision if and only if G has a vertex v of degree at least 5 and a circuit of size at least 5 which does not contain v.*

Proof. Suppose G is 3-connected with no internal 3-edge-cutset. The forward implication is trivial. Suppose then that G has a vertex v_0 of degree at least 5 and a circuit of size at least 5 which does not contain v_0.

By Lemma 2 G contains a W_4-subdivision centred on v_0. If $H = (V_H, E_H)$ is any W_4-subdivision centred on v_0, we write: C_H for the circuit of H which does not meet v_0; v_i^H, $1 \leqslant i \leqslant 4$, for the four vertices of degree 3 in H; and P_i^H for the path from v_0 to v_i^H in H (which meets C_H only at v_i^H), for each i, $1 \leqslant i \leqslant 4$. We assume (without loss of generality) that the v_i^H are arranged on C_H so that it is possible to move around C_H encountering v_1^H, v_2^H, v_3^H, v_4^H in exactly this order. Clearly

$N_G(v_0)\backslash N_H(v_0) \neq \emptyset$, since $\deg v_0 \geq 5$. If $u \in N_G(v_0)\backslash N_H(v_0)$, define the vertex set $U_H(u)$ as follows: If $u \notin V_H$, $U_H(u)$ is the bridge of $G \mid V_H$ which contains u; if $u \in V_H$, $U_H(u) = \{v_0, u\}$.

We will usually drop the superscripts and subscripts H when no ambiguity results.

There are three cases to consider.

Case 1. G has a W_4-subdivision H centred on v_0 and a vertex $u \in N_G(v_0)\backslash N_H(v_0)$ such that $U_H(u)$ contains a vertex u_1 of $C_H\backslash\{v_1^H, v_2^H, v_3^H, v_4^H\}$.

There is then a $v_0 - u_1$ path in $\langle U_H(u)\rangle$ which does not meet H except at v_0 and u_1, and this path together with H gives a W_5-subdivision in G. (See Fig. 2; the wavy line is the v_0–u_1 path in $\langle U_H(u)\rangle$.)

Case 2. G has a W_4-subdivision H centred on v_0 and a vertex $u \in N_G(v_0)\backslash N_H(v_0)$ such that $U_H(u)$ contains two vertices, other than v_0, on two separate P_i^H. (Note that in this case $u \notin V_H$.)

Let these two vertices be u_1, u_2. Since G is 3-connected there are two paths Q_1 and Q_2 in $\langle U(u)\rangle$ from u to u_1 and u_2 respectively which meet H only at u_1 and u_2 and which are vertex-disjoint except at u. Assume without loss of generality that u_1 is on P_1 and u_2 is on P_2 or P_3.

Subcase 2*a*: $\{u_1, u_2\} \neq \{v_1, v_3\}$.

Assume without loss of generality that $u_2 \neq v_3$. G contains a W_5-subdivision formed from H as follows: add the vertex u, the edge uv_0 and paths Q_1 and Q_2, and remove the internal vertices of the $v_1 - v_2$ path in C_H which avoids v_3. Figs. 3(a) and 3(b) illustrate this when u_2 is on P_2 or P_3 respectively, and in each case

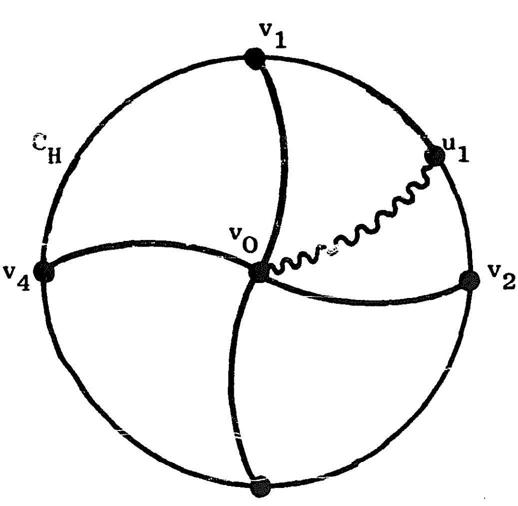


Fig. 2. Diagram for Case 1.

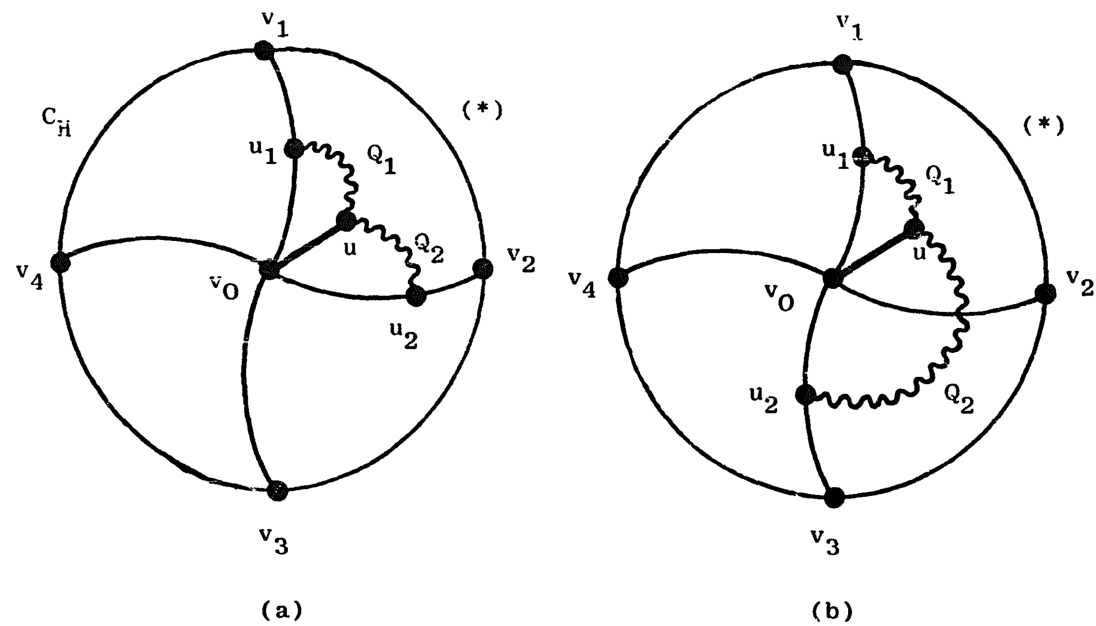


Fig. 3. Diagrams for Subcase 2a.

the starred portion $(*)$ is the path to be removed as just described. The reader may easily verify the presence of a W_5-subdivision in each instance.

Subcase 2b: $u_1 = v_1$, $u_2 = v_3$.

Put $W = \{v_0, v_1, v_3\}$. We may suppose that $U(u) \cap V_H = W$, since if $(U(u) \cap V_H) \backslash W \neq \emptyset$ then either Case 1 or Subcase 2a applies and we know then that G has a W_5-subdivision.

Suppose there is a path Q in G from an internal vertex w_1 of P_1 or P_3 to a vertex w_2 of H not on P_1 or P_3 where Q does not meet H except at its endpoints. Suppose without loss of generality that w_1 is on P_1. Then G contains a W_5-subdivision: Fig. 4 shows essentially all the configurations of w_1, w_2 and Q relative to H, and in each case removing the starred portion leaves a W_5-subdivision. (Note that Q avoids $U(u)$ by definition of the latter.) From now on suppose there is no such path Q. Thus, if $i \in \{1, 3\}$, then P_i is either a single edge or is not in the same bridge of $G \mid W$ as any vertex in $H - P_1 - P_3$.

Now let H_2 and H_4 be the components of $H - P_1 - P_3$ which contain v_2 and v_4

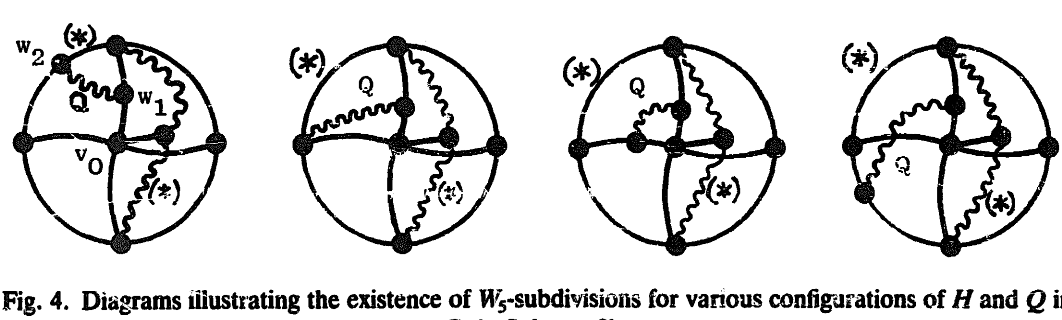


Fig. 4. Diagrams illustrating the existence of W_5-subdivisions for various configurations of H and Q in G, in Subcase 2b.

respectively. Thus H_2 (respectively, H_4) consists of the path $P_2[v_2, v_0)$ ($P_4[v_4, v_0)$) and the path obtained by removing the endpoints of the v_1–v_3 path in C which meets v_2 (v_4). Suppose there exists a path Q in G from a vertex w_1 of H_2 to a vertex w_2 of H_4 which avoids H except at its endpoints. Then once again it can be seen that G contains a W_5-subdivision: Fig. 5 shows the possible configurations, and in each case removing the starred portion leaves a W_5-subdivision. (Note that Q avoids $U(u)$.) From now on then suppose there is no such path Q. Thus H_2, H_4 are in different bridges of $G \mid W$, and we denote these bridges by U_2 and U_4 respectively.

We have so far shown that there are at least 3 different bridges of $G \mid W$, namely U_2, U_4 and $U(u)$, which do not contain any internal vertices (if such exist) of P_1 or P_3.

Now W is contained in a bridge of $G \mid W$ (namely any of U_2, U_4, $U(u)$); it follows that each bridge of $G \mid W$ contains some vertex of G not in W, and further that each bridge of $G \mid W$ contains W for otherwise 3-connectivity is violated. Now $G-v_0$ has a circuit D of size at least 5. If $V(D)$ is contained in some bridge of $G \mid W$ then that bridge trivially contains at least two vertices not in W. On the other hand, if $V(D)$ is not contained in any bridge of $G \mid W$, then D must meet v_1 and v_3 and the two bridges of $D \mid \{v_1, v_3\}$ are each contained in a bridge of $G \mid W$. But since D has length at least 5, some bridge of $D \mid \{v_1, v_3\}$ has at least two vertices other than v_1 and v_3. Thus there is a bridge U^* of $G \mid W$ which contains at least two vertices not in W.

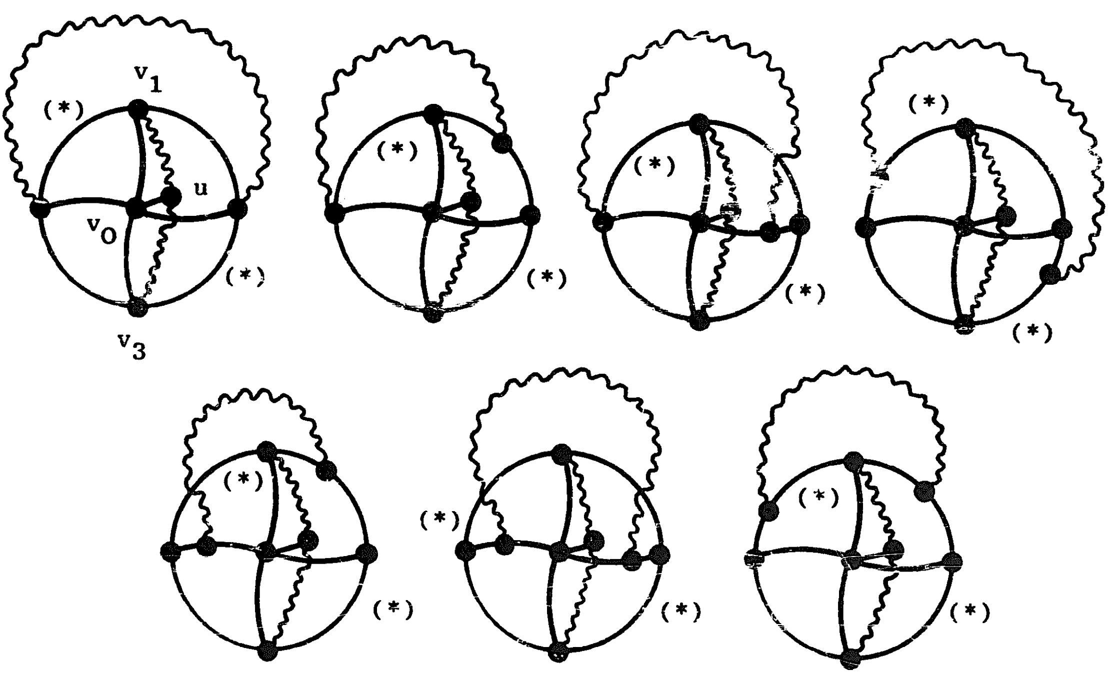


Fig. 5. Diagrams illustrating the existence of a W_5-subdivision when there is a H_2–H_4 path, in Subcase 2b.

Consider the set of edges

$$E' = \{xy \mid x \in W, y \in U^*\backslash W\}.$$

Clearly E' is a cutset of G. Since each bridge of $G \mid W$ contains W, we know $|E'| \geq 3$. Now one component of $G-E'$ contains $U^*\backslash W$, which we know has at least two vertices, and the other component of $G-E'$ contains W; thus each component of $G-E'$ contains at least two vertices. This implies that if $|E'| = 3$ then E' is an internal 3-edge-cutset, contradictory to our assumptions. Hence $|E'| \geq 4$. Thus some member x of W has at least two neighbours, say w_1 and w_2, in $U^*\backslash W$. Put $\{y, z\} = W\backslash\{x\}$. By 3-connectivity there exist two vertex-disjoint paths Q_y, Q_z in $\langle U^* \rangle$, avoiding x, from w_1, w_2 respectively to $\{y, z\}$. Furthermore there is a path R in $\langle U^* \rangle$ from some vertex w_1' on Q_y to some vertex w_2' on Q_z which avoids W and which meets Q_y, Q_z only at its endpoints.

Suppose for the moment that U^* meets at least one of P_1 and P_3 internally. (Note that then U^* is distinct from U_2, U_4 and $U(u)$, which cannot meet P_1 or P_3 internally as we observed earlier.) Now of course $U(u)$ contains the vertex u (which is not in W) and the three paths Q_1, Q_2 and uv_0 from u to $\{x, y, z\}$ which are vertex-disjoint except at u. We also note the existence of an x–y path S_{xy} in U_2 and an x–z path S_{xz} in U_4. The paths Q_y, Q_z, R, Q_1, Q_2, S_{xy} and S_{xz} together with the edges xw_1, xw_2 and uv_0 constitute a W_5-subdivision, in G, centred on x (see Fig. 6; the paths marked by a dagger are Q_1, Q_2 and uv_0 in some order). Note that it is not necessarily true that $x = v_0$, so this W_5-subdivision need not be centred on v_0.

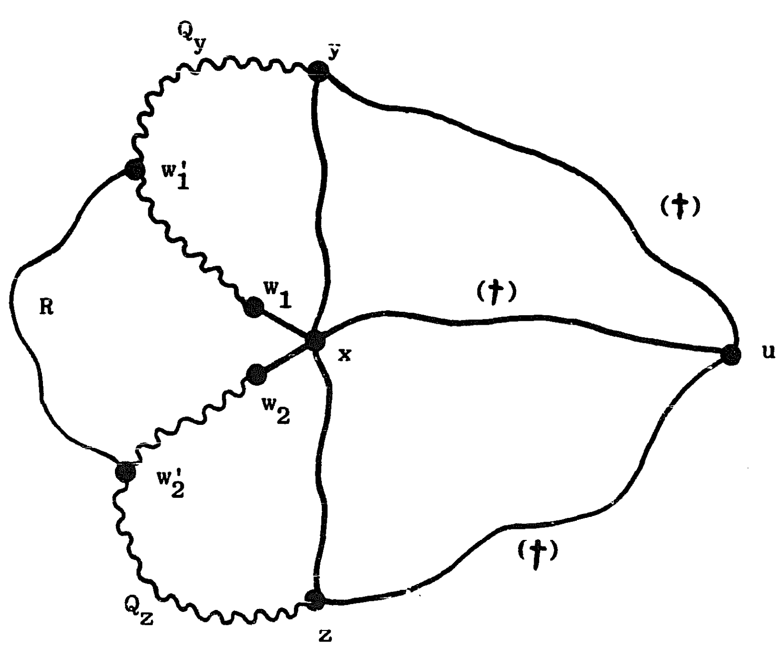


Fig. 6. A W_5-subdivision found in Subcase 2b.

Finally suppose that U^* does not meet P_1 or P_3 internally. Here U^* might conceivably be one of U_2, U_4 or $U(u)$, but in any event there certainly exist two bridges $U^{(1)}$, $U^{(2)}$ of $G \mid W$ other than U^*. For each $i \in \{1, 2\}$ we also know that $\langle U^{(i)} \rangle$ contains a vertex $u^{(i)}$ not in W and three paths $R_0^{(i)}$, $R_1^{(i)}$, $R_2^{(i)}$ from $u^{(i)}$ to v_0, v_1, v_3 respectively which are vertex-disjoint except at $u^{(i)}$. Now if $x = v_0$ then G contains the W_5-subdivision centred on v_0 which consists of the paths Q_y, Q_z, R, P_1, P_3, $R_0^{(1)}$, $R_1^{(1)}$ and $R_2^{(1)}$ and the edges v_0w_1, v_0w_2, as shown in Fig. 7(a). If $x \neq v_0$, suppose without loss of generality that $x = v_1$ and observe that then G contains the W_5-subdivision centred on v_0 consisting of the paths Q_y, Q_z, R, P_1, $R_0^{(1)}$, $R_1^{(1)}$, $R_2^{(1)}$, $R_1^{(2)}$ and $R_2^{(2)}$ and the edges v_1w_1, v_1w_2, as shown in Fig. 7(b). We remark that this W_5-subdivision is not centred on v_0, but rather on v_1.

Case 3. G has a W_4-subdivision H centred on v_0 and a vertex $u \in N_G(v_0) \setminus N_H(v_0)$ such that

$$U_H(u) \cap V_H \subseteq P_i^H \qquad \text{for some } i \in \{1, 2, 3, 4\}.$$

Assume without loss of generality that $U_H(u) \cap V_H \subseteq P_1^H$. Let $u_1(H, u)$ be the last vertex of $U_H(u) \cap V_H$ encountered in traversing P_1^H from V_0 to v_1^H. Assume that H, u are chosen to minimize

$$d_{P_1^H}(u_1(H, u), v_1^H).$$

Let Q_0 be a path in $\langle U_H(u) \rangle$ from v_0 to $u_1(H, u)$ which is disjoint from H other than at its endpoints, and put $Q_1 = P_1^H[v_0, u_1(H, u)]$. (See Fig. 8.) Set

$$A = (U_H(u) \cup V(Q_1)) \setminus \{v_0, u_1(H, u)\},$$
$$B = V(H - Q_1).$$

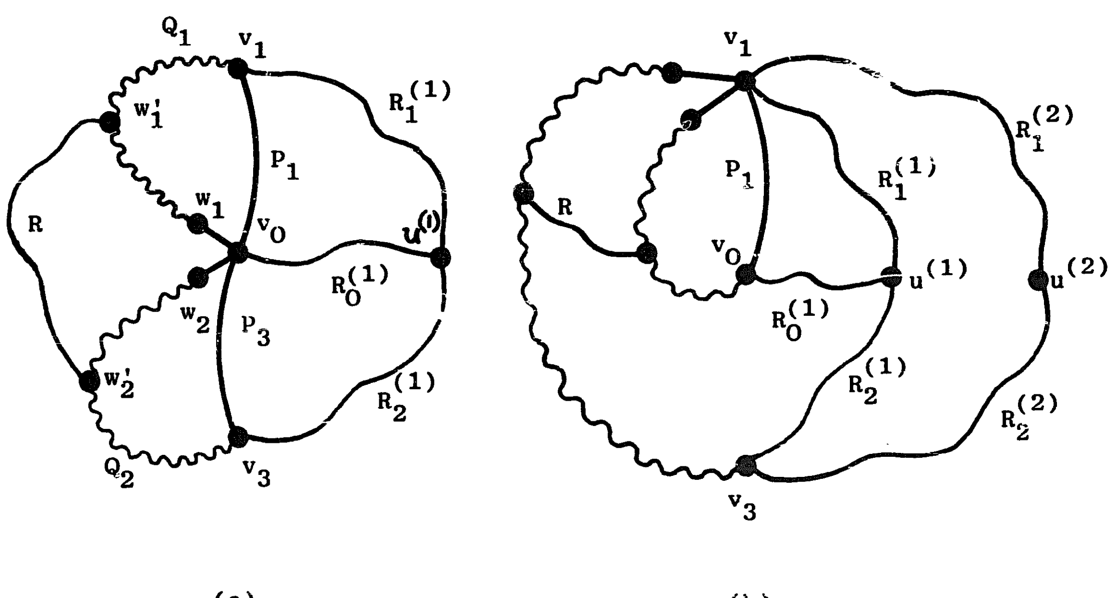


Fig. 7. W_5-subdivisions found in the final cases dealt with in Subcase 2b.

Observe that: $A \cap B = \emptyset$; $A \neq \emptyset$, for otherwise Q_0 and Q_1 are parallel edges; and $B \neq \emptyset$. Now the removal of $\{v_0, u_1(H, u)\}$ cannot disconnect A from B, since G is 3-connected. Hence there is a path Q_2 from A to B which avoids $\{v_0, u_1(H, u)\}$ and which, it may be assumed, meets A and B each only once, in vertices v_A and v_B respectively. Suppose $v_A \in U_H(u)\backslash V(Q_1)$. Then Q_2 is contained in $\langle U_H(u)\rangle$ which implies $v_B \in U_H(u)$, a contradiction. Thus

$$v_A \in V(Q_1)\backslash\{v_0, u_1(H, u)\}.$$

(See Fig. 8.) It follows that the distance from v_0 to $u_1(H, u)$ along Q_1 is at least 2.

Let $L = (V_L, E_L)$ be the W_4-subdivision obtained from H by replacing P_1^H with the v_0–v_1^H path, which we call P_1^L, consisting of Q_0 together with $P_1^H[u_1(H, u), v_1^H]$. We set $P_2^L = P_2^H$, $P_3^L = P_3^H$, $P_4^L = P_4^H$, $C_L = C_H$, so by definition $v_i^L = v_i^H$ for all $i \in \{1, 2, 3, 4\}$. Define u' to be the vertex adjacent to v_0 on Q_1; it

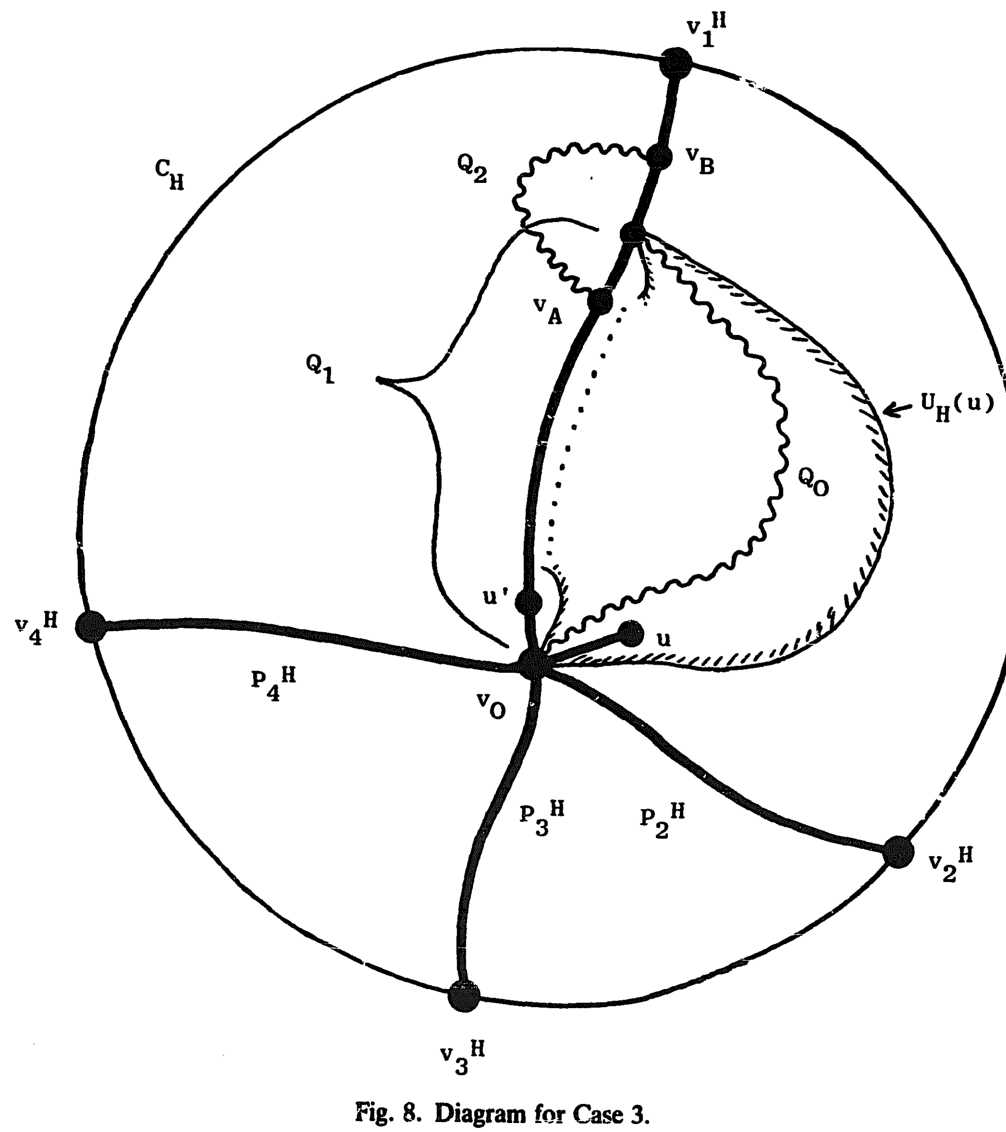


Fig. 8. Diagram for Case 3.

is not in L, by the last sentence of the previous paragraph, so $u' \in N_G(v_0)\backslash N_L(v_0)$. Clearly $v_B \in U_L(u')$. If $v_B \in C_L\backslash\{v_1^L, v_2^L, v_3^L, v_4^L\}$ then L, u' satisfy the hypotheses of Case 1 and so G has a W_5-subdivision. If $v_B \in V(P_2^L) \cup V(P_3^L) \cup V(P_4^L)$ then L, u' satisfy the hypotheses of Case 2 and so G has a W_5-subdivision. Thus we suppose that $v_B \in V(P_1^L)$. Since v_B is in B, it is on $P_1^L(u_1(H, u), v_1^L] = P_1^H(u_1(H, u), v_1^H]$. Thus $u_1(L, u')$, the last vertex on P_1^L (going away from v_0) in $U_L(u') \cap V_L$, is on $P_1^L(u_1(H, u), v_1^L]$. Thus

$$d_{P_1^L}(u_1(L, u'), v_1^L) < d_{P_1^L}(u_1(H, u), v_1^L) = d_{P_1^H}(u_1(H, u), v_1^H),$$

contradicting the minimality of $d_{P_1^H}(u_1(H, u), v_1^H)$ (with respect to H, u).

It is not difficult to see that the three cases we have considered are exhaustive: in fact, for every H and every $u \in N_G(v_0)\backslash N_H(v_0)$, $U_H(u)$ must be of one of the three types described. We have shown that in each case G must contain a W_5-subdivision, so the proof is complete. □

4. Algorithms

Theorem 3.1 forms the basis of the following polynomial time algorithm for solving SHP(W_4).

Algorithm 1.

Step 1. Input: Graph G.
Step 2. If $|V(G)| \leq 4$, reject G.
Step 3. If G is 3-connected, go to 4; otherwise:
- *3.1.* Find a separating set V_0 for G of at most two vertices, with V_0 minimal.
- *3.2.* If $|V_0| = 2$, form G' by adding an edge between the two members of V_0 if none exists already. If $|V_0| \leq 1$, set $G' = G$.
- *3.3.* Find the bridges $U_1, \ldots, U_k$ of $G' \mid V_0$.
- *3.4.* Apply the algorithm recursively to each $\langle U_i \rangle$, $1 \leq i \leq k$. If any $\langle U_i \rangle$ is accepted, accept G. Otherwise, reject G.

Step 4. If G has a vertex of degree at least 4, accept G; otherwise, reject G.

Algorithm 1 accepts a graph G if and only if one of the following holds:

(i) G is 3-connected and has a vertex of degree at least 4.

(ii) G has a separating set V_0 with $|V_0| \leq 2$ such that some subgraph induced by one of the bridges of $G' \mid V_0$ is accepted when the algorithm is recursively applied to it.

If a graph G has a separating set V_0 of size at most 2 then G has a W_4-subdivision if and only if some subgraph of G' induced by a bridge of $G' \mid V_0$

has a W_4-subdivision. This follows from the 3-connectivity of W_4. In effect, the only way a W_4-subdivision can "straddle" V_0 is in the case $|V_0| = 2$, by a single path which passes through V_0 twice, returning to the bridge of $G' \mid V_0$ that it came from; this is reflected in the addition of an edge between the two members of V_0 in Step 3.2.

This observation, together with Theorem 3.1, implies that one of (i), (ii) holds if and only if G has a W_4-subdivision. Thus Algorithm 1 solves SHP(W_4). It is clear also that the algorithm runs in polynomial time (it is a standard divide-and-conquer method).

We now present our algorithm for solving SHP(W_5), which employs a similar technique.

Algorithm 2.

Step 1. Input: Graph G.
Step 2. If $|V(G)| \leq 5$, reject G.
Step 3. If G is 3-connected, go to 4; otherwise, do Steps 3.1 to 3.4 which we do not write out as they are exactly the same as the corresponding steps in Algorithm 1 (although in 3.4 "the algorithm" must now be taken to refer to our algorithm here).
Step 4. If G has no internal 3-edge-cutset, go to 5; otherwise:
  4.1. Find an internal 3-edge-cutset E' of G. Suppose $E' = \{e_1, e_2, e_3\}$.
  4.2. Let G_1, G_2 be the components of $G-E'$. For each j, $1 \leq j \leq 3$, let the endpoints of e_j in G_1, G_2 be u_j, v_j respectively. Form G_1' from G_1 (respectively G_2' from G_2) by adding a single new vertex w_1 (w_2) adjacent to each of u_1, u_2, u_3 (v_1, v_2, v_3).
  4.3. Apply the algorithm recursively to G_1' and G_2'. If either is accepted, accept G; otherwise, reject G.
Step 5. If G has no vertex of degree at least 5, reject G; otherwise, continue.
Step 6. For each vertex v of G of degree at least 5, determine whether $G-v$ has a circuit of length at least 5. If some such $G-v$ has such a circuit, accept G; otherwise, reject G.

It is straightforward to use Theorem 3.4 to verify that this algorithm recognizes SHP(W_5). The argument is very similar to our justification of Algorithm 1, except that the case in which G has an internal 3-edge-cutset has to be dealt with as well. This is not very difficult, however; a W_5-subdivision H in G can only contain edges of such a cutset E' if either (i) some path of H leaves and returns to one component of $G-E'$ via E', or (ii) H contains one vertex of degree 3 in a different component of $G-E'$ from its other vertices of degree at least 3 (see Fig. 9). The way we form G_1' and G_2' fully takes account of these possibilities, and so it is not difficult to see that G has a W_5-subdivision if and only if at least one of G_1', G_2' does.

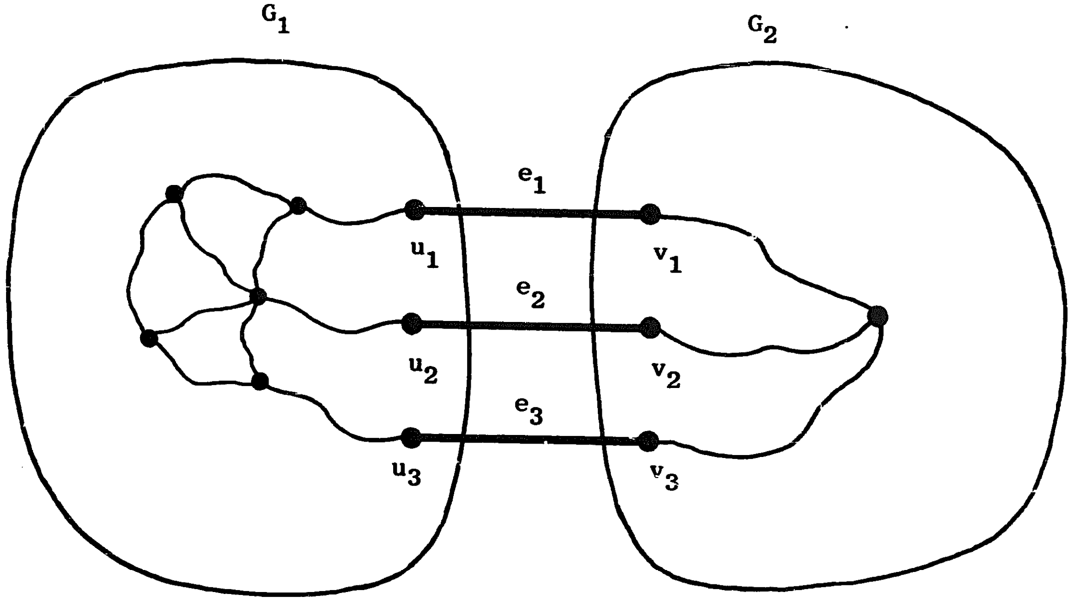


Fig. 9.

It is routine to verify that Algorithm 2 runs in polynomial time, provided that in Step 6 we use a polynomial time algorithm to determine whether a graph has a circuit of length at least 5. Such an algorithm certainly exists: one could use a divide-and-conquer method, similar to Algorithm 1, based on the fact that a 2-connected graph G with at least 5 vertices has no circuit of length ≥ 5 if and only if G has a pair of vertices such that every edge of G is incident with a vertex in the pair (see for example [9]).

5. Discussion

In going from the proof of Lemma 3.2, concerning W_4-subdivisions, to the proof of Theorem 3.4, concerning W_5-subdivisions, a considerable jump in difficulty occurs. As can possibly be appreciated, dealing with the exclusion of W_6-subdivisions (at least by our sort of approach) becomes even more complicated. There are a number of reasons for this. Firstly, an approach like ours would exploit the existence under suitable conditions of a W_5-subdivision H and would investigate how the bridges of $G \mid V(H)$ interact with H. We would expect to have rather more cases to deal with than the four cases we had using this kind of approach when H was a W_4-subdivision in the proof of Theorem 3.4. Secondly, in proving Theorem 3.4 we made great use of the fact that G not only had a W_4-subdivision, but a W_4-subdivision centred on v_0, our chosen vertex of degree at least 5. In proving a similar theorem for W_6 we would expect to exploit the presence in a graph satisfying suitable conditions of a W_5-subdivision. However, we may not be able to expect to find a W_5-subdivision centred exactly where we

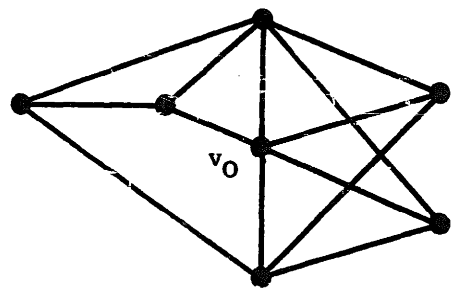


Fig. 10.

please. Certainly a graph G satisfying the conditions of Theorem 3.4 need not have a W_5-subdivision centred on a given vertex v_0 of degree at least 5. An example illustrating this possibility is given in Fig. 10; here there is no W_5-subdivision centred on v_0.

Thus, an attack on W_6 using our sort of approach can be expected to involve a prohibitive amount of case analysis. It would still be of interest to find a similar result for W_6, and to find a better approach to dealing with such excluded subdivisions for this and other pattern graphs.

References

[1] G.A. Dirac, A property of 4-chromatic graphs and remarks on critical graphs, J. London Math. Soc. 27 (1952) 69–81.
[2] R.J. Duffin, Topology of series-parallel networks, J. Math. Anal. Appl. 10 (1965) 303–318.
[3] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (W.H. Freeman and Co., San Francisco, 1979).
[4] D.W. Hall, A note on primitive skew curves, Bull. Amer. Math. Soc. 49 (1943) 935–937.
[5] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).
[6] A.S. LaPaugh and R.L. Rivest, The subgraph homeomorphism problem, in: Proc. 10th Ann. ACM Symp. on Theory of Comput. (Assoc. for Comput. Mach., New York, 1978) 40–50.
[7] K. Menger, Zur allgemeinen Kurventheorie, Fund. Math. 10 (1927) 96–115.
[8] N. Robertson and P.D. Seymour, Graph minors—a survey, in: Ian Anderson, ed., Surveys in Combinatorics 1985, London Mathematical Society Lecture Note Series 103 (Cambridge University Press, Cambridge, 1985) 153–171.
[9] P.D. Seymour and P.N. Walton, Detecting matroid minors, J. London Math. Soc. (2) 23 (1981) 193–203.